COMPLEMENTS OF CONNECTED HYPERSURFACES IN S^4

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ABSTRACT. Let X and Y be the complementary regions of a closed hypersurface M in $S^4 = X \cup_M Y$. We use the Massey product structure in $H^*(M;\mathbb{Z})$ to limit the possibilities for $\chi(X)$ and $\chi(Y)$. We show also that if $\pi_1(X) \neq 1$ then it may be modified by a 2-knot satellite construction, while if $\chi(X) \leq 1$ and $\pi_1(X)$ is abelian then $\beta_1(M) \leq 4$ or $\beta_1(M) = 6$. Finally we use TOP surgery to propose a characterization of the simplest embeddings of $F \times S^1$.

A closed hypersurface in S^n is orientable and has two complementary components, by the higher-dimensional analogue of the Jordan Curve Theorem. There have been sporadic papers presenting restrictions on the orientable 3-manifolds which may embed in S^4, but little is known about how many distinct embeddings there may be. (Here and in what follows, "embed" shall mean "embed as a TOP locally flat submanifold", unless otherwise qualified.) While the question of which rational homology 3-spheres embed smoothly in S^4 has received considerable attention, work on embeddings of more general 3-manifolds is very limited. Most of the relevant papers known to us are cited in [1].

The complementary components of embeddings of S^3 in S^4 are balls, by the Brown-Mazur-Schoenflies Theorem. A result of Aitchison shows that every embedding of $S^2 \times S^1$ in S^4 has one complementary component homeomorphic to $S^2 \times D^2$ [24]. The other component is a 2-knot complement, with Euler characteristic $\chi = 0$ and fundamental group a 2-knot group, and so embeddings of $S^2 \times S^1$ in S^4 correspond to 2-knots. But for 3-manifolds M with $\beta = \beta_1(M) > 1$ even the possible Euler characteristics of the complementary components are not known.

We consider here $\chi(W)$ and $\pi_1(W)$, for W the closure of a component of $S^4 \setminus M$. Our examples mostly involve Seifert fibred 3-manifolds M, and the embeddings are constructed from 0-framed "bipartedly slice" links [defined below] representing M. The obstructions to embeddings derive from the lower central series for $\pi_1(M)$ and its dual manifestation in terms of (Massey) products of classes in $H^1(M;\mathbb{Q})$.

In §1 we use the Mayer-Vietoris sequence and Poincaré-Lefshetz duality to show that if $S^4 = X \cup_M Y$ then $\chi(X) \equiv \chi(Y) \equiv 1 + \beta \ mod\,(2)$, and that we may assume that $1 - \beta \leq \chi(X) \leq 1 \leq \chi(Y) \leq 1 + \beta$. All such possibilities may be realised by embeddings of $\#^\beta(S^2 \times S^1)$, and all except for $1-\beta$ by embeddings of $T_g \times S^1$. In §3 we use the Massey product structure in $H^*(M;\mathbb{Z})$ to show that if M fibres over an orientable base surface and the fibration has Euler number 1 then $\chi(X) = \chi(Y) = 1$ is the only possibility. At the other extreme, $\chi(X) = 1 - \beta$ is realizable only if the rational nilpotent completion of $\pi_1(M)$ is that of the free group $F(\beta)$.

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In §4 we give a criterion for a complementary region to be aspherical and of cohomological dimension at most 2. We then show in §5 that we may use a "satellite" construction based on 2-knots to modify the fundamental group of a complementary component which is not 1-connected, without changing the other complementary component. In §6 we show that $\pi_1(X)$ can be abelian only if $\beta \leq 4$ or $\beta = 6$, and then $\pi_1(X)$ is one of Z/nZ, $\mathbb{Z} \oplus Z/nZ$, $\mathbb{Z}^2$ or $\mathbb{Z}^3$. We give examples realizing these possibilities. In §7 we assume that M is Seifert fibred, with orientable base orbifold. If the generalized Euler invariant ε_S is 0 and $\chi(X) < 0$ then the regular fibre has nonzero image in $H_1(Y;\mathbb{Q})$, and so $\chi(X) > 1-\beta$. If $\varepsilon_S \neq 0$ then $\chi(X) = \chi(Y) = 1$.

When $M = F \times S^1$ or when M is the total space of an S^1-bundle with non-orientable base the simplest embeddings of M have one complementary component $X \simeq F$ and the other with cyclic fundamental group. In §8 we sketch how surgery may be used to identify such embeddings (up to s-cobordism). (No such argument is yet available when M fibres over an orientable base with Euler number 1.)

1. Euler characteristic and cup product

Let M be a closed connected orientable 3-manifold with fundamental group π, and let $\beta = \beta_1(M;\mathbb{Q})$. Let T_M be the torsion subgroup of $H_1(M;\mathbb{Z})$ and $\ell_M : T_M \times T_M \to \mathbb{Q}/\mathbb{Z}$ the torsion linking pairing.

Lemma 1. *Suppose M embeds in S^4, and let X and Y be the closures of the components of $S^4 \setminus M$. Then $\chi(X) + \chi(Y) = 2$, $\chi(X) \equiv \chi(Y) \equiv 1 + \beta$ mod (2), and $1 - \beta \leq \chi(X) \leq 1 + \beta$.*

Proof. The Mayer-Vietoris sequence for $S^4 = X \cup_M Y$ gives isomorphisms

$$H_i(M;\mathbb{Z}) \cong H_i(X;\mathbb{Z}) \oplus H_i(Y;\mathbb{Z}),$$

for $i = 1, 2$, while $H_j(X;\mathbb{Z}) = H_j(Y;\mathbb{Z}) = 0$ for $j > 2$. Hence $\chi(X) + \chi(Y) = 2$. Moreover, $H_2(X;\mathbb{Z}) \cong H^1(Y;\mathbb{Z})$, by Poincaré-Lefshetz duality. Let $\gamma = \beta_1(X)$. Then $\beta_2(X) = \beta - \gamma$, so $\chi(X) = 1 + \beta - 2\gamma$, where $0 \leq \gamma \leq \beta$. □

Clearly $\chi(X)$ is determined by $\pi_1(X)$, and conversely $\chi(X)$ determines the rank of $H_1(X;\mathbb{Z})$. One of the subsidiary themes of this paper is that $\chi(X)$ can have deeper influence on $\pi_1(X)$. See Lemma 3 below, for instance.

We may assume X and Y are chosen so that $\chi(X) \leq \chi(Y)$. Thus if $\beta = 0$ then $\chi(X) = \chi(Y) = 1$, while if $\beta = 1$ then $\chi(X) = 0$ and $\chi(Y) = 2$.

Let j_X and j_Y be the inclusions of M into X and Y, and let T_X and T_Y be the torsion subgroups of $H_1(X;\mathbb{Z})$ and $H_1(Y;\mathbb{Z})$, respectively. Then $T_M \cong T_X \oplus T_Y$, and each of these summands is self-annihilating under ℓ_M, by Poincaré-Lefshetz duality. Hence ℓ_M is hyperbolic [16]. In particular, $T_Y \cong Ext(T_X, \mathbb{Z}) \cong Hom(T_X, \mathbb{Q}/\mathbb{Z})$, and so T_M is a direct double: it is (non-canonically) isomorphic to $T_X \oplus T_X$.

The cohomology ring $H^*(M;\mathbb{Z})$ is determined by the 3-fold product

$$\mu_M : \wedge^3 H^1(M;\mathbb{Z}) \to H^3(M;\mathbb{Z})$$

and Poincaré duality. If we identify $H^3(M;\mathbb{Z})$ with $\mathbb{Z}$ we may view μ_M as an element of $\wedge^3(H_1(M;\mathbb{Z})/T_M)$. Every finitely generated free abelian group H and linear homomorphism $\mu : \wedge^3 H \to \mathbb{Z}$ is realized by some closed orientable 3-manifold [26]. (If $\beta \leq 2$ then $\wedge^3 \mathbb{Z}^\beta = 0$, and so $\mu_M = 0$.)

Lemma 2. *The cup product 3-form μ_M is 0 if and only if all cup products of classes in $H^1(M;\mathbb{Z})$ are 0. Its restrictions to each of $\wedge^3 H^1(X;\mathbb{Z})$ and $\wedge^3 H^1(Y;\mathbb{Z})$ are 0.*

Proof. Poincaré duality implies immediately that $\mu_M = 0$ if and only if all cup products from $\wedge^2 H^1(M;\mathbb{Z})$ to $H^2(M;\mathbb{Z})$ are 0.

Since $H^3(X;\mathbb{Z}) = H^3(Y;\mathbb{Z}) = 0$, the restrictions of μ_M to $\wedge^3 H^1(X;\mathbb{Z})$ and $\wedge^3 H^1(Y;\mathbb{Z})$ are 0. $\square$

See [19] for the parallel case of doubly sliced knots.

If $\mu_M \neq 0$ then $H^1(X;\mathbb{Z})$ and $H^1(Y;\mathbb{Z})$ must be nontrivial proper summands. However, if $\mu_M = 0$ this lemma places no condition on these summands.

2. BIPARTEDLY SLICE LINKS AND S^1-BUNDLE SPACES

Any closed orientable 3-manifold M may be obtained by integrally framed surgery on some r-component link L in S^3, with $r \geq \beta$. We may assume that the framings are even [15], and then after adjoining copies of the 0-framed Hopf link Ho (i.e., replacing M by $M\#S^3 \cong M$) we may modify L so that it is 0-framed. (If the component L_i has framing $2k \neq 0$ we adjoin $|k|$ disjoint copies of Ho and band-sum L_i to each of the $2k$ new components, with appropriately twisted bands.)

If $L = L_+ \cup L_-$ is the union of an s-component slice link L_+ and an $(r-s)$-component slice link L_- then ambient surgery on S^3 in S^4 shows that M embeds in S^4, with complementary components having $\chi = 1 + 2s - r$ and $1 - 2s + r$. (We shall say that such a link is *bipartedly sliceable*.) In particular, if L is a slice link then $\beta = r$ and there are embeddings realizing each value of $\chi(X)$ allowed by this lemma, including one with a 1-connected complementary region. (However, it is not clear that every closed hypersurface in S^4 derives from a 0-framed bipartedly sliceable link.)

Each component of $S^4 \setminus M$ has a natural Kirby-calculus presentation, with 1-handles represented by dotting the components of one part of L and 2-handles represented by the remaining components of L. Hence its fundamental group has a presentation with generators corresponding to the meridians of the dotted circles and relators corresponding to the remaining components.

For instance, $\#^\beta(S^2 \times S^1)$ is the result of 0-framed surgery on the β-component trivial link, and so has embeddings realizing all the possibilities for Euler characteristics allowed by Lemma 1. In particular, it has an embedding with complementary regions $X \cong \natural^\beta(D^3 \times S^1)$ and $Y \cong \natural^\beta(S^2 \times D^2)$. (In this case $\mu_M = 0$.)

Let T be the torus, $T_g = \#^g T$ the closed orientable surface of genus $g \geq 0$, and $P_c = \#^c RP^2$ the closed non-orientable surface with $c \geq 1$ cross-caps. If $p : E \to F$ is an S^1-bundle with base a closed surface F and orientable total space E then $\pi_1(F)$ acts on the fibre via $w = w_1(F)$, and such bundles are classified by an Euler class $e(p)$ in $H^2(F;\mathbb{Z}^w) \cong \mathbb{Z}$. If we fix a generator $[F]$ for $H_2(F;\mathbb{Z}^w)$ we may define the Euler number of the bundle by $e = e(p)([F])$. (We may change the sign of e by reversing the orientation of E.) Let $M(g;(1,e))$ and $M(-c;(1,e))$ be the total spaces of the S^1-bundles with base T_g and P_c (respectively), and Euler number $-e$. (This is consistent with the notation for Seifert fibred 3-manifolds in §5 below.)

Suppose first that F is orientable. Then $E = M(g;(1,e))$ can only embed in S^4 if $e = 0$ or ± 1, since $T_E = 0$ if $e = 0$ and is cyclic of order e otherwise. The 3-torus $M(1;(1,0)) \cong S^1 \times S^1 \times S^1$ may be obtained by 0-framed surgery on the Borromean rings $Bo = 6^3_2$. (We refer to the tables of [23].) Since $M(g;(1,0)) \cong T_g \times S^1$ is an iterated fibre sum of copies of $T \times S^1$, it may be obtained by 0-framed surgery on a $(2g+1)$-component link L which shares some of the Brunnian properties of Bo. It has an embedding as the boundary of $T_g \times D^2$, the regular neighbourhood

of the unknotted embedding of T_g in S^4, with the other complementary region having fundamental group $\mathbb{Z}$. It is easy to see that if $g \geq 1$ then $T_g \times S^1$ has other embeddings with $\chi(X)$ realizing each even value $> 1-\beta$. On the other hand, $\mu_{T_g \times S^1} \neq 0$, and so no embedding has a complementary region Y with $\beta_1(Y) = 0$.

Changing the framing on one component of Bo to 1, and applying a Kirby move to isolate this component gives the disjoint union of the Whitehead link $Wh = 6^2_3$ and the unknot. Since the linking numbers are 0 the framings are unchanged, and we may delete the isolated 1-framed unknot. Thus $M(1;(1,1))$ may be obtained by 0-framed surgery on Wh. The corresponding modification of the standard 0-framed $(2g+1)$-component link L representing $T_g \times S^1$ involves changing the framing of the component L_{2g+1} whose meridian represents the central factor of π. Performing a Kirby move and deleting an isolated 1-framed unknot gives a 0-framed $2g$-component link representing $M(g;(1,1))$. Since the original link had partitions into two trivial links with $g+1$ and g components respectively, the new link has a partition into two trivial g-component links. However this is the only partition into slice sublinks, for as we shall see in §3 consideration of the Massey product structure shows that all embeddings of $M(g;(1,1))$ have $\chi(X) = \chi(Y) = 1$.

Suppose now that F is nonorientable. Then $M(-c;(1,e))$ embeds if and only if it embeds as the boundary of a regular neighbourhood of an embedding of P_c with normal Euler number e [3]. We must have $e \leq 2c$ and $e \equiv 2c \ mod\ (4)$. The standard embedding of RP^2 in S^4 is determined up to composition with a reflection of S^4. The complementary regions are each homeomorphic to a disc bundle over RP^2 with normal Euler number 2, and so have fundamental group $Z/2Z$. The standard embeddings of P_c are obtained by taking iterated connected sums of these building blocks $\pm(S^4, RP^2)$, and in each case the exterior has fundamental group $Z/2Z$. The regular neighbourhoods of P_c are disc bundles with boundary $M(-c;(1,e))$. Thus $M(-c;(1,e))$ has an embedding with one complementary component $X_{c,e}$ a disc bundle over P_c and the other component $Y_{c,e}$ having fundamental group $Z/2Z$.

The constructions in the appendix to [3] suggest framed link presentations for $M(-c;(1,e))$. The standard embedding corresponds to a 0-framed $(c+1)$-component link assembled from copies of the $(2,4)$-torus link 4^2_1 and its reflection. This is the union of an unknot and a trivial c-component link, but has no other partitions into slice links. However, we can do better if we recall that $P_c \cong P_{c-2g} \# T_g$ for any g such that $2g < c$. Using copies of $\pm 4^2_1$ and Bo accordingly, for each $e \leq 2c$ such that $e \equiv 2c \ mod\ (4)$ we find a representative link with partitions into trivial sublinks corresponding to all the values $2 - c \leq \chi(X) \leq \min\{2 - \frac{|e|}{2}, 1\}$ such that $\chi(X) \equiv c \ mod\ (2)$. (Note Figure A.3 of [3].) Are any other values realized? In particular, does $M(-3;(1,6))$ embed with $\chi(X) = \chi(Y) = 1$?

If we move beyond the class of S^1-bundle spaces, we may give an example of "intermediate" behaviour. It is not hard to show that if $H \cong \mathbb{Z}^\beta$ with $\beta \leq 5$ then for every $\mu : \wedge^3 H \to \mathbb{Z}$ there is an epimorphism $\lambda : H \to \mathbb{Z}$ such that μ is 0 on the image of $\wedge^3 \mathrm{Ker}(\lambda)$. Hence there are splittings $H \cong A \oplus B$ with A of rank 3 or 4 such that μ restricts to 0 on each of $\wedge^3 A$ and $\wedge^3 B$. However if $\beta = 6$ this fails for

$$\mu = e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_5 \wedge e_6 + e_2 \wedge e_4 \wedge e_5.$$

(Here $\{e_i\}$ is the basis for $Hom(H, \mathbb{Z})$ which is Kronecker dual to the standard basis of $H \cong \mathbb{Z}^6$.) For every epimorphism $\lambda : \mathbb{Z}^6 \to \mathbb{Z}$ there is a rank 3 direct summand A of $\mathrm{Ker}(\lambda)$ such that μ is nontrivial on $\wedge^3 A$. [This requires a little

calculation. Suppose that $\lambda = \Sigma\lambda_i e_i^*$. If $\lambda_6 \neq 0$ then we may take A to be the direct summand containing $\langle f_1, f_2, f_3\rangle$, where $f_j = \lambda_6 e_j - \lambda_j e_6$, for $1 \leq j \leq 3$, for then $\mu(f_1 \wedge f_2 \wedge f_3) = \lambda_6^3 \neq 0$. Similarly if λ_3 or λ_4 is nonzero. If $\lambda_3 = \lambda_4 = \lambda_6 = 0$ but $\lambda_1 \neq 0$ then we may take A to be the direct summand containing $\langle g_2, e_4, g_5\rangle$, where $g_2 = \lambda_1 e_2 - \lambda_2 e_1$ and $g_5 = \lambda_1 e_5 - \lambda_5 e_1$. Similarly if λ_2 or λ_5 is nonzero.]

This example arose in a somewhat different context [5]. It is the cup product 3-form of the 3-manifold M given by 0-framed surgery on the 6-component link of Figure 6.1 of [5]. This link has certain "Brunnian" properties. All the 2-component sublinks, all but three of the 3-component sublinks and six of the 4-component sublinks are trivial. Thus M has embeddings in S^4 with $\chi(X) = -1$ or 1, corresponding to partitions of L into a pair of trivial sublinks, but there are no embeddings with $\chi(X) = -5$ or -3, since μ_M does not satisfy the second assertion of Lemma 2.

3. MASSEY PRODUCTS AND LOWER CENTRAL SERIES

Massey product structures in the cohomology of M provide further obstructions to finding embeddings with given $\chi(X)$. For instance, if $H^2(X;\mathbb{Q}) \cong \mathbb{Q}$ or 0 then all triple Massey products $\langle a, b, c\rangle$ of elements $a, b, c \in H^1(X;\mathbb{Q})$ are proportional.

The Massey product structures for classes in $H^1(X;\mathbb{F})$, with $\mathbb{F}$ a prime field $\mathbb{Q}$ or $\mathbb{F}_p$, are closely related to the rational and p-lower central series of the fundamental group of $\pi_1(X)$ (see [7]). We shall let $G_{[n]}$ denote the nth term of the descending lower central series of a group G, defined inductively by $G_{[1]} = G$ and $G_{[n+1]} = [G, G_{[n]}]$, for all $n \geq 1$. Similarly, the rational lower central series is given by letting $G^{\mathbb{Q}}_{[1]} = G$ and $G^{\mathbb{Q}}_{[k+1]}$ be the preimage in G of the torsion subgroup of $G/[G, G^{\mathbb{Q}}_{[k]}]$. Then $G/G^{\mathbb{Q}}_{[k]}$ is a torsion free nilpotent group, and $\{G^{\mathbb{Q}}_{[k]}\}_{k\geq 1}$ is the most rapidly descending series of subgroups of G with this property.

The $\mathbb{N}il^3$-manifold $M = M(1;(1,1))$ has fundamental group $\pi \cong F(2)/F(2)_{[3]}$, with a presentation

$$\pi = \langle x, y, z \mid z = xyx^{-1}y^{-1},\ xz = zx,\ yz = zy\rangle.$$

Every element of π has an unique normal form $x^m y^n z^p$. The images X, Y of x, y in $H_1(\pi;\mathbb{Z}) \cong H_1(T;\mathbb{Z})$ form a (symplectic) basis. Let ξ, η be the Kronecker dual basis for $H^1(\pi;\mathbb{Z})$. Define functions ϕ_ξ, ϕ_η and $\theta : \pi \to \mathbb{Z}$ by

$$\phi_\xi(x^m y^n z^p) = \frac{m(1-m)}{2},\ \phi_\eta(x^m y^n z^p) = \frac{n(1-n)}{2} \text{ and } \theta(x^m y^n z^p) = -mn - p,$$

for all $x^m y^n z^p \in \pi$. (We consider these as inhomogeneous 1-cochains with values in the trivial π-module $\mathbb{Z}$.) Then

$$\delta\phi_\xi(g, h) = \xi(g)\xi(h), \quad \delta\phi_\eta(g, h) = \eta(g)\eta(h) \quad \text{and} \quad \delta\theta(g, h) = \xi(g)\eta(h),$$

for all $g, h \in \pi$. Thus $\xi^2 = \eta^2 = \xi \cup \eta = 0$, and the Massey triple products $\langle \xi, \xi, \eta\rangle$ and $\langle \xi, \eta, \eta\rangle$ are represented by the 2-cocycles $\phi_\xi\eta + \xi\theta$ and $\theta\eta + \xi\phi_\eta$, respectively. On restricting these to the subgroups generated by $\{x, z\}$ and $\{y, z\}$, we see that they are linearly independent.

In fact, $\langle \xi, \xi, \eta\rangle \cup \eta$ and $\langle \xi, \eta, \eta\rangle \cup \xi$ each generate $H^3(\pi;\mathbb{Z})$ (i.e., these Massey products are the Poincaré duals of Y and X, respectively). This is best seen topologically. Let $p : M \to T$ be the natural fibration of M over the torus, and let x and y be simple closed curves in T which represent a basis for $\pi_1(T) \cong \mathbb{Z}^2$. The group $H_2(M;\mathbb{Z}) \cong \mathbb{Z}^2$ is generated by the images of the fundamental classes of the tori $T_x = p^{-1}(x)$ and $T_y = p^{-1}(y)$. If we fix sections in M for the loops x and y we

see that $[T_x] \bullet x = [T_y] \bullet y = 0$ while $|[T_x \bullet y| = |T_y \bullet x| = 1$. Hence $[T_x]$ and $[T_y]$ are Poincaré dual to η and ξ, respectively. Since $\langle \xi, \xi, \eta \rangle$ restricts nontrivially to T_x and trivially to T_y we must have $\langle \xi, \xi, \eta \rangle \cup \eta \neq 0$, and similarly $\langle \xi, \eta, \eta \rangle \cup \xi \neq 0$.

Since the components of Wh are unknotted M embeds in S^4, with $\chi(X) = \chi(Y) = 1$, and since $\beta = 2$ we have $\mu_M = 0$. On the other hand, M has no embedding with $\chi(X) = -1$, for otherwise $H^3(X;\mathbb{Z})$ would contain $\langle \xi, \xi, \eta \rangle \cup \eta$, and so be nontrivial.

A similar strategy may be used for $M = M(g;(1,1))$ and $\pi = \pi_1(M)$, when $g > 1$. Let $\{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ be the basis for $H = H^1(\pi;\mathbb{Z})$ which is Kronecker dual to a symplectic basis for $H_1(\pi;\mathbb{Z}) \cong H_1(F;\mathbb{Z})$. Then $H = A \oplus B$, where A and B are self-annihilating with respect to cup product on F. The Massey triple products $\langle \alpha_i, \alpha_i, \beta_i \rangle$ and $\langle \alpha_i, \beta_i, \beta_i \rangle$ (for $1 \leq i \leq g$) form a basis for $H^2(\pi;\mathbb{Z})$ which is Poincaré dual to the given basis for $H_1(\pi;\mathbb{Z})$. If $L \leq H$ is a direct summand of rank $> g$ then there are $a \in L \cap A$ and $b \in L/A$ such that $a \cup b \neq 0$ in $H^2(F;\mathbb{Z})$. We may assume that $a = \alpha_1$ and then $b = \beta_1 + b'$, where b' is in the span of $\{\alpha_2, \beta_2, \dots, \alpha_g, \beta_g\}$. But then $\langle a, a, b \rangle \cup b \neq 0$. It follows that if $j : M \to S^4$ is any embedding then $H^1(X;\mathbb{Z})$ and $H^1(Y;\mathbb{Z})$ each have rank at most g, and so $\chi(X) = \chi(Y) = 1$.

The 3-form μ_M is 0 if and only if $\pi/\pi^{\mathbb{Q}}_{[3]} \cong F(\beta)/F(\beta)^{\mathbb{Q}}_{[3]}$ [26]. However, this is a rather weak condition. The next lemma gives a stronger result.

Lemma 3. *If $H_1(Y;\mathbb{Z}) = 0$ then $\pi/\pi_{[k]} \cong F(\beta)/F(\beta)_{[k]}$, for all $k \geq 1$.*

Proof. If $H_1(Y;\mathbb{Z}) = 0$ then $H_2(X;\mathbb{Z}) = 0$, and T must be 0, by the non-degeneracy of ℓ_M, so $H_1(M;\mathbb{Z}) \cong H_1(X;\mathbb{Z}) \cong \mathbb{Z}^\beta$. Let $f : \vee^\beta S^1 \to X$ be any map such that $H_1(f;\mathbb{Z})$ is an isomorphism. Then j_X and f induce isomorphisms on all quotients of the lower central series, by Stallings' Theorem [25], and so $\pi/\pi_{[k]} \cong F(\beta)/F(\beta)_{[k]}$, for all $k \geq 1$. □

If M is the result of surgery on a β-component slice link L then it has an embedding with a 1-connected complementary region, and so this lemma applies. However there are slice links which are not homology boundary links. (See Figure 8.1 of [9].) For such links the abelianization of the link group does not factor through a homomorphism onto a free group.

There are parallel results for the rational lower central series and the p-central series, for primes p, with coefficients $\mathbb{Q}$ and $\mathbb{F}_p$, respectively. In particular, if $\beta_1(Y) = 0$ then $\pi/\pi^{\mathbb{Q}}_{[k]} \cong F(\beta)/F(\beta)^{\mathbb{Q}}_{[k]}$, for all $k \geq 1$. Stallings' Theorem can be refined to relate "freeness" of quotients of such series and the vanishing of higher Massey products [7]. For instance, the kernel of cup product $\cup_G$ from $\wedge^2 H^1(G;\mathbb{Q})$ to $H^2(G;\mathbb{Q})$ is isomorphic to $G^{\mathbb{Q}}_{[2]}/G^{\mathbb{Q}}_{[3]}$ ([26] – see also §12.2 of [9].) In particular, $\cup_G$ is injective if $G_{[2]}/G_{[3]}$ is finite.

Unfortunately, the fact that $\mathrm{Ker}(\cup_X) \subseteq \mathrm{Ker}(\cup_M)$ does not have useful consequences for M. For if $\beta_1(X) < \beta$ then $\mathrm{Ker}(\cup_X)$ has rank at most $\binom{\beta_1(X)}{2} \leq \binom{\beta-1}{2} = \binom{\beta}{2} - \beta$, which is a lower bound for the rank of $\mathrm{Ker}(\cup_M)$. If $\beta_1(X) = \beta$ then $\beta_2(X) = 0$ so $\mu_M = 0$, and all cup products of degree-1 classes are 0.

4. DIMENSION AND FUNDAMENTAL GROUP

Since the complementary regions are 4-manifolds with non-empty boundary they are homotopy equivalent to 3-dimensional complexes. However, when such a space

is homotopically 2-dimensional remains an open question, in general. We shall say that $c.d.W \leq n$ if the equivariant chain complex of the universal cover $\widetilde{W}$ is chain homotopy equivalent to a complex of projective $\mathbb{Z}[\pi_1(W)]$-modules of length $\leq n$.

Theorem 4. *Let W be a component of $S^4 \setminus M$, where M is a closed hypersurface. Then $c.d.W \leq 2$ if and only if $\pi_1(j_W)$ is an epimorphism. If so, then W is aspherical if and only if $c.d.\pi_1(W) \leq 2$ and $\chi(W) = \chi(\pi_1(W))$.*

Proof. Let $\Gamma = \mathbb{Z}[\pi_1(W)]$, and let $C_* = C_*(\widetilde{W};\mathbb{Z})$ be the chain complex of $\widetilde{W}$, considered as a complex of free left Γ-modules. Then $H_i(W;\Gamma) = H_i(C_*)$ is $H_i(\widetilde{W};\mathbb{Z})$, with the natural Γ-module structure, for all i. The equivariant cohomology of $\widetilde{W}$ is defined in terms of the cochain complex $C^* = Hom_\Gamma(C_*,\Gamma)$, which is naturally a complex of right modules. Let $\overline{C}^q$ be the left Γ-module obtained via the canonical anti-involution of Γ, defined by $g \mapsto g^{-1}$ for all $g \in \pi_1(W)$, and let $H^j(W;\Gamma) = H^j(\overline{C}^*)$. Equivariant Poincaré-Lefshetz duality gives isomorphisms $H_i(W;\Gamma) \cong H^{4-i}(W,\partial W;\Gamma)$ and $H^j(W;\Gamma) \cong H_{4-j}(W,\partial W;\Gamma)$, for all $i,j \leq 4$.

If $c.d.W \leq 2$ then $H_i(\widetilde{W},\partial\widetilde{W};\mathbb{Z}) = 0$ for $i \leq 1$, and so $\partial\widetilde{W}$ is connected. Therefore $\pi_1(j_W)$ must be surjective. Conversely, if $\pi_1(j_W)$ is an epimorphism then we may assume that W may be obtained from M (up to homotopy) by adjoining cells of dimension ≥ 2. Hence $H_i(W,\partial W;\Gamma)$ and $H^j(W,\partial W;\Gamma)$ are 0 for $i,j \leq 1$. Therefore $H_q(W;\Gamma) = H^q(W;\Gamma) = 0$ for all $q > 2$, and so C_* is chain homotopy equivalent to a complex P_* of finitely generated projective Γ-modules of length at most 2, by Wall's finiteness criteria [29].

If W is aspherical then $c.d.\pi_1(W \leq 2$, and we must have $\chi(W) = \chi(\pi_1(W))$. Conversely, if $\pi_1(j_W)$ is onto then $\Pi = H_2(P_*) \cong \pi_2(W)$ is the only obstruction to asphericity. If, moreover, $c.d.\pi_1(W) \leq 2$ we may apply Schanuel's Lemma, to see that P_* splits as

$$P_* = \Pi \oplus (Z_1 \to P_1 \to P_0),$$

where π is concentrated in degree 2, Z_1 is the submodule of 1-cycles and $Z_1 \to P_1 \to P_0$ is a resolution of the augmentation module $\mathbb{Z} = H_0(P_*)$. Now $\mathbb{Z} \otimes_\Gamma \Pi \cong H_2(W;\mathbb{Z})$ is a free abelian group of rank $\chi(W) - \chi(\pi_1(W))$. If, moreover, $\chi(W) = \chi(\pi_1(W))$ then $\Pi = 0$, and so W is aspherical, since the weak Bass Conjecture holds for groups of cohomological dimension ≤ 2 [8]. $\square$

In our applications of Theorem 4 below, $\pi_1(W)$ is either free, free abelian or the fundamental group of an aspherical surface. Hence all projective Γ-modules are stably free, and so we could use an old result of Kaplansky instead of invoking [8]. There seems to be no simple criterion for W to be aspherical when $c.d.W = 3$.

Let K be the Artin spin of a nontrivial classical knot, and let $X = X(K)$ be the exterior of a tubular neighbourhood of K in S^4. Then $\pi_1(X) \cong \pi K$, the knot group, and $M = \partial X \cong S^2 \times S^1$. In this case $c.d.\pi K = 2$ and $\chi(X) = \chi(\pi K) = 0$, but $\pi_1(j_X)$ is not onto, and X is not aspherical. (Thus $c.d.X = 3$.)

There are two essentially different partitions of the standard link representing $T_g \times S^1$ into moieties with $g+1$ and g components. For one, $X \cong S^1 \times (\natural^g(D^2 \times S^1)$, which is aspherical (as to be expected from Theorem 4); for the other, $\pi_1(X) \cong \mathbb{Z}^2 * F(g-1)$, and X is not aspherical. (In neither case is Y aspherical.)

5. Modifying the group

We may modify embeddings by "2-knot surgery" on a complementary region, as follows. Let N_γ be a regular neighbourhood in X of a simple closed curve representing $\gamma \in \pi_1(X)$. Then $\overline{S^4 \setminus N_\gamma} \cong S^2 \times D^2$ contains Y and M. If K is a 2-knot with exterior $E(K)$ then $\Sigma = \overline{S^4 \setminus N_\gamma} \cup E(K)$ is a homotopy 4-sphere, and so is homeomorphic to S^4. The complementary components to M in Σ are $X_{\gamma,K} = \overline{X \setminus N_\gamma} \cup E(K)$ and Y. This construction applies equally well to simple closed curves in Y.

When $M = S^2 \times S^1$ is embedded as the boundary of a regular neighbourhood of the trivial 2-knot, with $X = D^3 \times S^1$ and $Y = S^2 \times D^2$, the core $S^2 \times \{0\} \subset Y_1$ is K, realized as a satellite of the trivial knot. This construction gives all possible embeddings of $S^2 \times S^1$ in S^4 (up to composition with self-homeomorphisms of domain and range), by Aitchison's result [24]. For this reason, we shall refer to this construction as the 2*-knot satellite* construction.

Let t be the image of a meridian for K in the knot group $\pi K = \pi_1(E(K))$. If γ has infinite order in $\pi_1(X)$ then $\pi_1(X_{\gamma,K})$ is a free product with amalgamation $\pi_1(X) *_{\mathbb{Z}} \pi K$; if it has finite order c then $\pi_1(X_{\gamma,K}) \cong \pi_1(X) *_{Z/cZ} (\pi K/\langle\langle t^c \rangle\rangle)$. (Note that if $K = \tau_c k$ is a nontrivial twist spin then $\pi K/\langle\langle t^c\rangle\rangle \cong \pi K' \rtimes Z/cZ$.)

If $\gamma = 1$ then any simple closed curve representing γ is isotopic to one contained in a small ball, since homotopy implies isotopy for curves in 4-manifolds. Hence in this case 2-knot surgery does not change the topology of X.

It is well known that a nilpotent group with cyclic abelianization is cyclic. It follows that the natural projection of $\pi_1(X_{\gamma,K})$ onto $\pi_1(X)$ induces isomorphisms of corresponding quotients by terms of the lower central series. Thus we cannot distinguish these groups by such quotients. Nevertheless, we have the following result.

Theorem 5. *If $\pi_1(X) \neq 1$ then there are infinitely many groups of the form $\pi_1(X_{\gamma,K})$.*

Proof. Suppose first that $\pi_1(X)$ is torsion-free and that $\gamma \neq 1$. If $\pi_1(K) \cong Z/nZ \rtimes \mathbb{Z}$ then $\pi_1(X_{\gamma,K}) \cong \pi_1(X) *_{\mathbb{Z}} \pi K$ is an extension of a torsion-free group by the free product of countably many copies of Z/nZ. Since $Z/nZ \rtimes \mathbb{Z}$ is the group of the 2-twist spin of a 2-bridge knot, for every odd n, the result follows.

If $\pi_1(X)$ has an element γ of finite order $c > 1$ then we use instead Cappell-Shaneson 2-knots. Let a be an integer, and let $f_a(t) = t^3 - at^2 + (a-1)t - 1$. If $a > 5$ the roots α, β and γ of f_a are real, and we may assume that $\gamma < \beta < \alpha$. Elementary estimates give the bounds

$$\frac{1}{a} < \gamma < \frac{1}{2} < \beta < 1 - \frac{1}{a} < a - 2 < \alpha < a.$$

If $A \in SL(3,\mathbb{Z})$ is the companion matrix of f_a then $\mathbb{Z}^3 \rtimes_A \mathbb{Z}$ is the group of a "Cappell-Shaneson" 2-knot K. The quotient $\mathbb{Z}^3/(A^c - I)\mathbb{Z}^3$ is a finite group of order the resultant $Res(f_a(t), t^c - 1) = (\alpha^c - 1)(\beta^c - 1)(\gamma^c - 1)$, where α, β and γ are the roots of $f_a(t)$. This simplifies to

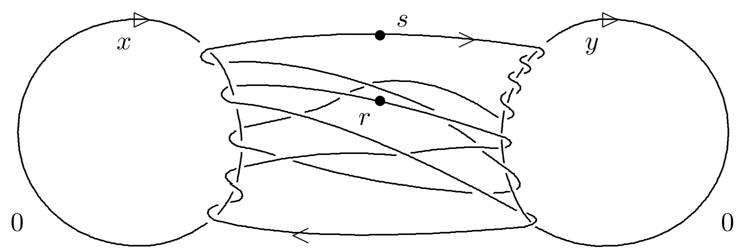
$$\alpha^p + \beta^p + \gamma^p - (\alpha\beta)^p - (\beta\gamma)^p - (\gamma\alpha)^p = \alpha^p(1 - \beta^p - \gamma^p) + \varepsilon,$$

where $0 < \varepsilon < 2$. It follows easily from our estimates that $|Res(f_a(t), t^c - 1)| > a^{c-1}$, if $a > 3c$. Hence $\pi K/\langle\langle t^c\rangle\rangle$ is a finite group of order $> ca^{c-1}$. We then use the fact that finitely presentable groups have an essentially unique representation as the

fundamental group of a graph of groups, with all vertex groups finite or one ended. (See Proposition 7.4 of Chapter IV of [4].) Thus if K and L are two such 2-knots such that $\pi K/\langle\langle t^c\rangle\rangle$ and $\pi L/\langle\langle t^c\rangle\rangle$ are finite groups of different orders, both greater than that of any of the finite vertex groups in such a representation of $\pi_1(X)$ then $\pi_1(X_{\gamma,K}) \ncong \pi_1(X_{\gamma,L})$. $\square$

If $H_1(M;\mathbb{Z}) \neq 0$ then X is not simply-connected, and so there are infinitely many embeddings with one complementary region Y and distinguishable by the fundamental groups of the other region, by Theorem 5. However if M is an homology 3-sphere then X and Y are homology balls, and it may not be easy to decide whether $\pi_1(X)$ and $\pi_1(Y)$ are nontrivial. When $M = S^3$ the complementary regions are homeomorphic to the 4-ball D^4, by the Brown-Mazur-Schoenflies Theorem. If $\pi_1(M) \neq 1$ is there an homology 4-ball X with $M \cong \partial X$, $\pi_1(X) \neq 1$ and the normal closure of the image of $\pi_1(M)$ in $\pi_1(X)$ being the whole group? If so, there is an embedding with one complementary region X and the other 1-connected.

Perhaps the simplest nontrivial example of a smooth embedding of an homology 3-sphere with neither complementary region 1-connected is given by the link displayed below. For this example $\pi_1(X) \cong \pi_1(Y) \cong I^*$, the binary icosahedral group, with presentation $\langle x, y \mid x^{-2}yxy,\ y^{-4}xyxy\rangle$.



If we swap the 0-framings and the dots, we obtain a Kirby-calculus presentation for Y. Since the loops r, s, x and y determine words $x^{-2}yxy$, $y^{-4}xyx$, $srsr^{-2}$ and $s^{-4}rsr$, respectively, $\pi_1(X)$ and $\pi_1(Y)$ have equivalent presentations. (There are 32 possible choices for the crossings involving only the dotted curves, all giving similar examples. Is there a choice for which there is a homeomorphism of S^3 interchanging the images of L_- and L_+?)

Other examples of this kind may be found in [20]. Lickorish showed also that any two groups G, H with balanced finite presentations and isomorphic abelianizations are the fundamental groups of a pair of complementary regions of some connected hypersurface in S^4 [21]. In particular, any two perfect groups with balanced presentations can be realized as $\pi_1(X)$ and $\pi_1(Y)$ for some embedding of an homology 3-sphere in S^4.

Are there optimal "minimal" embeddings of M, for given $\chi(X)$? For instance, is there an embedding for which the natural map $j_\Delta : \pi \to \pi_1(X) \times \pi_1(Y)$ is onto? This is clearly so if both factors are nilpotent, since $H_1(j_\Delta)$ is an isomorphism, and so j_Δ induces epimorphisms on all corresponding quotients of the lower central series. However these quotients are rarely isomorphic.

Theorem 6. *If* $\pi/\pi^{\mathbb{Q}}_{[3]} \cong (\pi_1(X)/\pi_1(X)^{\mathbb{Q}}_{[3]}) \times (\pi_1(Y)/\pi_1(Y)^{\mathbb{Q}}_{[3]})$ *then* $\chi(X) = 1 - \beta$ *or* $3 - \beta$.

Proof. Let $\gamma = \beta_1(X) \geq \frac{\beta}{2}$. If the 2-step quotients $(G/G^{\mathbb{Q}}_{[3]})$ are isomorphic then $\mathrm{Ker}(\cup_M)$ has rank at most $\binom{\gamma}{2} + \binom{\beta-\gamma}{2}$. Since $\beta_2(\pi) = \beta$ we must have

$$\binom{\beta}{2} - \beta \leq \binom{\gamma}{2} + \binom{\beta-\gamma}{2}.$$

This reduces to $\beta \geq \gamma(\beta - \gamma)$, and so either $\gamma \geq \beta - 1$ or $\beta = 4$ and $\gamma = 2$. In the latter case, consideration of μ_M shows that the rank of $\pi^{\mathbb{Q}}_{[2]}/\pi^{\mathbb{Q}}_{[3]}$ is at least $3 \neq \binom{2}{2} + \binom{2}{2}$, so this cannot occur. Thus $\chi(X) = 1 + \beta - 2\gamma \leq 3 - \beta$. $\square$

If j is any embedding with $H_1(Y;\mathbb{Z}) = 0$ (respectively, $\chi(X) = 1 - \beta$) then $H_2(j_\Delta)$ (respectively, $H_2(j_\Delta;\mathbb{Q})$ is an epimorphism, and so j_Δ induces isomorphisms on all quotients of the (rational) lower central series.

If F is a closed orientable surface then the embedding j of $M \cong F \times S^1$ as the boundary of a regular neighbourhood of the standard unknotted embedding of F in S^4 has $\chi(X) = 3 - \beta$ and j_Δ an isomorphism. On the other hand, if $\beta = 2$ then $\cup_M = 0$, by Poincaré duality for M, so $\pi^{\mathbb{Q}}_{[2]}/\pi^{\mathbb{Q}}_{[3]} \neq 0$. Therefore for no embedding j with $\chi(X) = 1$ is $H_2(j_\Delta;\mathbb{Q})$ an epimorphism. Can anything more be said about the cases with $\chi(X) = 3 - \beta$ (and β even)?

If $\pi_1(X)$ is a nontrivial proper direct factor of π then $\pi \cong \pi_1(F) \times \mathbb{Z}$ for some closed orientable surface F, and so $M \cong F \times S^1$. In this case, either $F = S^2$ and $\pi_1(X) \cong \mathbb{Z}$ or F is aspherical and $\pi_1(X) \cong \pi_1(F)$.

If $\pi_1(X)$ is a free factor of π then it is a 3-manifold group, and the image of the fundamental class $[M]$ in $H_3(\pi_1(X);\mathbb{Z})$ is trivial, since $M = \partial X$ and so $H_3(j_X) = 0$. Hence $\pi_1(X)$ is a free group. In particular, $\pi \cong \pi_1(X) * \pi_1(Y)$ only if π is itself a free group, and then $M \cong \#^\beta(S^2 \times S^1)$.

6. Abelian fundamental group

In this section we shall show that manifolds with embeddings for which $\pi_1(X)$ is abelian are severely constrained.

Theorem 7. *Suppose M has an embedding in S^4 for which $\pi_1(X)_{[2]} = \pi_1(X)_{[3]}$. Then either $\beta \leq 4$ or $\beta = 6$. If $\beta = 0$ or 2 then $\pi_1(X) \cong Z/nZ$ or $\mathbb{Z} \oplus Z/nZ$, respectively, for some $n \geq 1$, while if $\beta = 1$, 3, 4 or 6 then $\pi_1(X) \cong \mathbb{Z}^r$, where $r = \lfloor \frac{\beta+1}{2} \rfloor$. If $\pi_1(X)$ is abelian and $\beta = 1$ or 3 then X is aspherical.*

Proof. Let $r = \beta_1(X)$, $A = H_1(X;\mathbb{Z})$ and $\tau = T_X$. Then $2r \geq \beta$ and $A \cong \mathbb{Z}^r \oplus \tau$. Since A is abelian, $H_2(A;\mathbb{Z}) = A \wedge A \cong \mathbb{Z}^{\binom{r}{2}} \oplus \tau^r \oplus (\tau \wedge \tau)$.

If $\pi_1(X)_{[2]} = \pi_1(X)_{[3]}$ then $H_2(A;\mathbb{Z})$ is a quotient of $H_2(\pi_1(X);\mathbb{Z})$, by the 5-term exact sequence of low degree for $\pi_1(X)$ as an extension of A. This in turn is a quotient of $H_2(X;\mathbb{Z}) \cong \mathbb{Z}^{\beta-r}$, by Hopf's Theorem. Hence $\binom{r}{2} \leq \beta - r \leq r$, and so $r \leq 3$. If $\tau \neq 0$ then either $r = \beta = 0$ and $\tau \wedge \tau = 0$, or $r = 1$, $\beta = 2$ and $\tau \wedge \tau = 0$. In either case, τ is (finite) cyclic. If $\beta \neq 0$ or 2 then $\tau = 0$ and either $r = \beta = 1$, or $r = 2$ and $\beta = 3$ or 4, or $r = 3$ and $\beta = 6$. The final assertion follows immediately from Theorem 4. $\square$

If $\pi_1(X)$ is abelian, $r = \beta = 0$ and $T_M = 0$ then X is contractible. In the remaining cases X cannot be aspherical, since either $\pi_1(X)$ has nontrivial torsion (if $\beta = 0$), or $H_2(X;\mathbb{Z})$ is too big (if $\beta = 2$ or 4), or $H_3(X;\mathbb{Z})$ is too small (if $\beta = 6$).

If we assume merely that $\pi_1(X)_{[2]}/\pi_1(X)_{[3]}$ is finite (i.e., that the rational lower central series stabilizes after one step) then $\cup_X$ is injective, and a similar calculation gives the same restrictions on β.

Embeddings with $\pi_1(X)$ abelian realizing these possibilities may be easily found. (If $\pi_1(X) \neq 1$ then 2-knot surgery gives further examples with $\pi_1(X)$ nonabelian and $\pi_1(X)_{[2]} = \pi_1(X)_{[3]}$.) The simplest examples are for $\beta = 0, 1$ or 3, with $M \cong S^3$, $M = S^2 \times S^1$ or $S^1 \times S^1 \times S^1$ the boundary of a regular neighbourhood of a point or of the standard unknotted embedding of S^2 or T in S^4, respectively.

Other examples may be given in terms of representative links. When $\beta = 0$ the $(2, 2n)$ torus link gives examples with $X \cong Y$ and $\pi_1(X) \cong Z/nZ$. When $\beta = 1$ we may use any knot which bounds a slice disc $D \subset D^4$ such that $\pi_1(D^4 \setminus D) \cong \mathbb{Z}$, such as the unknot or the Kinoshita-Terasaka knot. (All such knots have Alexander polynomial 1. Conversely every Alexander polynomial 1 knot bounds a TOP locally flat slice disc with group $\mathbb{Z}$, by a striking result of Freedman.) The links 8^3_5 and 8^3_6 give further simple examples. (These each have a trivial 2-component sublink and an unknotted third component which represents a meridian of the first component or the product of meridians of the first two components, respectively.) When $\beta = 2$ any 2-component link with unknotted components and linking number 0, such as the trivial 2-component link or Wh, gives examples with $\pi_1(X) \cong \mathbb{Z}$. We may construct examples realizing $\mathbb{Z} \oplus Z/nZ$ from the 4-component link obtained from Bo by replacing one component by its $(2, 2n)$ cable. When $\beta = 3$ we may use the links Bo, 9^3_9 or 9^3_{18}. (These each have a trivial 2-component sublink and an unknotted third component which represents the commutator of the meridians of the first two components. However neither of the latter two links is Brunnian.)

Let L be the 4-component link obtained from Bo by adjoining a parallel to the third component, and let M be the 3-manifold M obtained by 0-framed surgery on L. Then the meridians of L represent a basis $\{e_i\}$for $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^4$, and $\mu_M = e_1 \wedge e_2 \wedge e_3 + e_1 \wedge e_2 \wedge e_4$. This link may be partitioned into the union of two trivial 2-component links in two essentially different ways, and ambient surgery gives two essentially different embeddings of M. If the sublinks are $\{L_1, L_2\}$ and $\{L_3, L_4\}$ then the complementary components have fundamental groups $\mathbb{Z}^2$ and $F(2)$. Otherwise, the complementary components are homeomorphic and have fundamental group $\mathbb{Z}^2$.

If M is an example with $\beta = 6$ and $\pi_1(X)$ and $\pi_1(Y)$ abelian then

$$\mu_M = e_1 \wedge e_5 \wedge e_6 + e_2 \wedge e_4 \wedge e_6 + e_3 \wedge e_4 \wedge e_5 + e_1 \wedge e_2 \wedge \tilde{e}_6 + e_1 \wedge e_3 \wedge \tilde{e}_5 + e_2 \wedge e_3 \wedge \tilde{e}_4,$$

where $\{e_1, e_2, e_3\}$ is a basis for $H_1(X; \mathbb{Z})$ and $\{e_4, e_5, e_6\}$ and $\{\tilde{e}_4, \tilde{e}_5, \tilde{e}_6\}$ are bases for $H_1(Y; \mathbb{Z})$. The simplest link giving rise to such a 3-manifold is a 6-component link with all 2-component sublinks trivial, a partition into two trivial 3-component links, and also a partition into two copies of Bo. It also has some trivial 4-component sublinks, but no trivial 5-component sublinks. We shall not give further details.

In all of the above examples except for one $\pi_1(Y)$ is also abelian. Note that Theorem 7 does *not* apply to $\pi_1(Y)$, as it uses the hypothesis $\beta_1(X) \geq \frac{1}{2}\beta$!

7. SEIFERT FIBRED 3-MANIFOLDS

We shall assume henceforth that M is Seifert fibred. Let $M = M(g; S)$ be the orientable Seifert fibred 3-manifold with base orbifold $T_g(\alpha_1, \dots, \alpha_r)$ and Seifert data $S = \{(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)\}$, where $1 < \alpha_i$ and $(\alpha_i, \beta_i) = 1$, for all $1 \leq i \leq r$.

If $c > 0$ we let also $M(-c;S)$ be the orientable Seifert fibred 3-manifold with base orbifold $\#^c RP^2(\alpha_1,\dots,\alpha_r)$ and Seifert data S. If $r=1$, we allow also the possibility $\alpha_1 = 1$. Let $\varepsilon_S = -\Sigma_{i=1}^{i=r}(\beta_i/\alpha_i)$ be the generalized Euler invariant of the Seifert bundle. (Our notation is based on that of [13]. In particular, we do not assume that $0<\beta_i<\alpha_i$.)

Let $p: M \to B$ be the projection to the base orbifold B, and let $|B|$ be the surface underlying B. If h is the image of the regular fibre in π then the subgroup generated by h is normal in π, and $\pi^{orb}(B) \cong \pi/\langle h\rangle$.

Lemma 8. [2] *Let M a an orientable Seifert fibred 3-manifold. If B is nonorientable or if $\varepsilon_S \neq 0$ then $H^*(M;\mathbb{Q}) \cong H^*(\#^\beta S^2 \times S^1;\mathbb{Q})$. Otherwise, the image of h in $H_1(M;\mathbb{Q})$ is nonzero, and $H^*(M;\mathbb{Q}) \cong H^*(|B| \times S^1;\mathbb{Q})$.*

Proof. There is a finite regular covering $q: \widehat{M} \to M$, which is an S^1-bundle space with orientable base $\widehat{B}$, say. Let $G = Aut(q)$. Then $H^*(M;\mathbb{Q}) \cong H^*(\widehat{M};\mathbb{Q})^G$. If B is nonorientable or if $\varepsilon_S \neq 0$ then the regular fibre has image 0 in $H_1(M;\mathbb{Q})$, and so $H^*(\widehat{B};\mathbb{Q})$ maps onto $H^*(M;\mathbb{Q})$. Hence all cup products of degree-1 classes are 0. In such cases, $H^*(M;\mathbb{Q}) \cong H^*(\#^\beta S^2 \times S^1;\mathbb{Q})$. Otherwise, $\widehat{M} \cong \widehat{B} \times S^1$ and G acts orientably on each of S^1 and $\widehat{B}$. Hence the image of h in $H_1(M;\mathbb{Q})$ is nonzero and $H^*(M;\mathbb{Q}) \cong H^*(|B| \times S^1;\mathbb{Q})$. □

We may use the observations on cup product from §1 to extract some information on the image of the regular fibre under the maps $H_1(j_X)$ and $H_1(j_Y)$.

Theorem 9. *Let $M = M(g;S)$ where $g \geq 1$ and $\varepsilon_S = 0$. If M embeds in S^4 then $\chi(X) > 1-\beta = -2g$ and $\chi(Y) < 1+\beta = 2g+2$. If $\chi(X) < 0$ then the image of h in $H_1(Y;\mathbb{Q})$ is nontrivial.*

Proof. Let $\{a_i^*, b_i^*; 1 \leq i \leq g\}$ be the images in $H^1(M;\mathbb{Q})$ of a symplectic basis for $H^1(|B|;\mathbb{Q})$. Then $a_i^*(h) = b_i^*(h) = 0$ for all i. Let $\theta \in H^1(M;\mathbb{Q})$ be such that $\theta(h) \neq 0$. By Lemma 8 we have

$$H^*(M;\mathbb{Q}) \cong H^*(|B| \times S^1;\mathbb{Q}) \cong \mathbb{Q}[\theta, a_i^*, b_i^*,\ \forall\ i \leq g]/I,$$

where I is the ideal $(\theta^2, a_i^{*2}, b_i^{*2}, \theta a_i^* b_i^* - \theta a_j^* b_j^*, a_i^* a_j^*, b_i^* b_j^*,\ \forall\ 1 \leq i < j \leq g)$.

Since $\theta a_1^* b_1^* \neq 0$ the triple product $\mu_M \neq 0$, and so M has no embedding with $\beta_2(Y) = 0$ (see §1). Hence $\chi(X) = 1-\beta$ ($\Leftrightarrow \chi(Y) = 1+\beta$) is impossible.

If $\chi(X) < 0$ then $\beta_1(X) > g+1$, and so the image of $H^1(X;\mathbb{Q})$ in $H^1(M;\mathbb{Q})$ must contain some pair of classes from the image of $H^1(|B|;\mathbb{Q})$ with nonzero product. But then it cannot also contain θ, since all triple products of classes in $H^1(X;\mathbb{Q})$ are 0. Thus the image of $H^1(Y;\mathbb{Q})$ must contain a class which is nontrivial on h, and so $j_Y(h) \neq 0$ in $H_1(Y;\mathbb{Q})$. □

In particular, if $g = 1$ then $\chi(X) = 0$ and $\chi(Y) = 2$.

Theorem 9 also follows from Lemma 3, since the centre of π is not contained in the commutator subgroup $\pi_{[2]} = [\pi,\pi]$.

If the base orbifold B is nonorientable or if $\varepsilon_S \neq 0$ then $\mu_M = 0$, by Lemma 8, and so the argument of Theorem 9 does not extend to these cases. However, Lemma 8 also suggests that when $\varepsilon_S \neq 0$ we should be able to use Massey product arguments as in §2 (where we considered the case $S = \emptyset$).

Theorem 10. *Let $M = M(g;S)$, where $g \geq 0$ and $\varepsilon_S \neq 0$. If M embeds in S^4 with complementary regions X and Y then $\chi(X) = \chi(Y) = 1$.*

Proof. The group $\pi = \pi_1(M(g;S))$ has a presentation

$$\langle x_1, y_1, \ldots, x_g, y_g, c_1, \ldots, c_r, h \mid \Pi[a_i, b_i]\Pi c_j = 1,\ c_i^{\alpha_i} h^{\beta_i} = 1,\ h\ \mathit{central}\rangle.$$

We may assume that $g \geq 1$, for if $g = 0$ then M is a $\mathbb{Q}$-homology 3-sphere and the result is clear. To calculate cup products and Massey products of pairs of elements of a standard basis for $H^1(\pi;\mathbb{Q})$ (corresponding to the Kronecker dual of a symplectic basis for $H_1(|B|;\mathbb{Q})$), it suffices to reduce to the case $g = 1$. Let $G = \pi/\langle\langle x_2, y_2, \ldots, x_g, y_g\rangle\rangle$, so G has a presentation

$$\langle x, y, c_1, \ldots, c_r, h \mid [x, y]\Pi c_j = 1,\ c_i^{\alpha_i} h^{\beta_i} = 1,\ h\ \mathit{central}\rangle.$$

Let $G_\tau = \langle\langle c_1, \ldots, c_r, h\rangle\rangle$, and let H be the preimage in G of the torsion subgroup of $G/[G, G_\tau]$. Then $G_\tau/H \cong \mathbb{Z}$, with generator t, say, and $[x, y] = t^e$ for some $e \neq 0$. Every element has a normal form $g = x^m y^n t^p w$, with $w \in H$. Define functions ϕ_ξ, ϕ_η and $\theta : \pi \to \mathbb{Q}$ by

$$\phi_\xi(x^m y^n t^p w) = \frac{m(1-m)}{2}, \quad \phi_\eta(x^m y^n t^p w) = \frac{n(1-n)}{2}$$

$$\text{and} \quad \theta(x^m y^n t^p w) = -mn - \frac{p}{e},$$

for all $x^m y^n t^p w \in G$. (In effect, we are passing to the $\mathbb{N}il^3$-group G/H, with presentation $\langle x, y, t \mid [x, y] = t^e,\ t\ \mathit{central}\rangle$.) We may now complete the argument as in §2, and we may conclude that only $\chi(X) = \chi(Y) = 1$ is possible when $\varepsilon_S \neq 0$. □

If $\chi(X) = 0$ and h has nonzero image in $H_1(X;\mathbb{Q})$ then S is skew-symmetric (i.e., the Seifert data occurs in pairs $\{(a, b), (a, -b)\}$), by the main result of [11]. (In particular, this must be the case if g and ε_S are 0.) Conversely, if S is skew-symmetric and all cone point orders a_i are odd then $M(0;S)$ embeds smoothly. Since $\beta = 1$ we must have $\chi(X) = 0$ and $H_1(Y;\mathbb{Q}) = 0$. (In fact, for the embedding constructed on page 693 of [3] the component X has a fixed point free S^1-action.) Hence also $M(g;S)$ embeds smoothly, as in Lemma 3.2 of [3], which gives embeddings with $\chi(X) = 0$. Is there a natural choice of 0-framed bipartedly sliceable link representing $M(0;S)$? If so then all values of $\chi(X)$ consistent with Theorem 8 are possible for $M(g;S)$.

However, even if $\chi(X) = 0$ the other hypothesis of the main theorem of [11] need not hold. For instance, we may partition the standard 0-framed link representing $M = T_2 \times S^1$ into 3- and 2-component trivial sublinks in two essentially different ways. For one, $\pi_1(X) \cong \mathbb{Z} \times F(2)$ and $\pi_1(Y) \cong F(2)$, while for the other $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}^2$ and $\pi_1(Y) \cong \mathbb{Z}^2$.

If ℓ_M is hyperbolic then all even cone point orders have the same 2-adic valuation, by Theorem 3.7 of [3] (when $g < 0$) and Lemma 6 of [12] (when $g \geq 0$).

Donald has stronger results for the case of smooth embeddings, using gauge theoretic methods rather than algebraic topology [6]. If $M(g;S)$ embeds smoothly and $\varepsilon_S = 0$ then S is skew-symmetric. (Thus if $\varepsilon_S = 0$ and all cone point orders are odd then $M(g;S)$ embeds smoothly if and only if S is skew-symmetric.) If $M(-c;S)$ (with $c > 0$) embeds smoothly then S is weakly skew-symmetric (i.e., the data occurs in pairs $\{(a, b), (a, -b')\}$, where $b' = b$ or $bb' \equiv 1 \bmod (a)$) and all even cone point orders are equal.

Are there further obstructions related to 2-torsion in the cone point orders of the base orbifolds B? What are the possible values of $\chi(X)$ for embeddings of $M(g;S)$ (with $\varepsilon_S = 0$) or $M(-c;S)$?

8. Recognizing the simplest embeddings

The simplest 3-manifolds to consider in the present context are perhaps the total spaces of S^1-bundles over surfaces. Most of those which embed have canonical "simplest" embeddings. We give some evidence that these may be characterized up to s-concordance by the conditions $\pi_1(X) \cong \pi_1(F)$, where F is the base, and $\pi_1(Y)$ is abelian. (Embeddings $j_0, j_1 : M \to S^4$ are s-concordant if they extend to an embedding of $M \times [0,1]$ in $S^4 \times [0,1]$ whose complementary regions are s-cobordisms *rel* ∂. We need this notion as it is not yet known whether 5-dimensional s-cobordisms are always products.)

Suppose first that $M \cong T_g \times S^1$. There is a canonical embedding $j_g : M \to S^4$, as the boundary of a regular neighbourhood of the standard smooth embedding $T_g \subset S^3 \subset S^4$. Let X_g and Y_g be the complementary components. Then $X_g \cong T_g \times D^2$ and $Y_g \simeq S^1 \vee \bigvee^{2g} S^2$, and so $\pi_1(Y_g) \cong \mathbb{Z}$.

We shall assume henceforth that $g \geq 1$, since embeddings of $S^2 \times S^1$ and $S^3 = M(0;(1,1))$ may be considered well understood. Let h be the image of the fibre in $\pi = \pi_1(E)$.

Lemma 11. *Let $j : T_g \times S^1 \to S^4$ be an embedding such that $\pi_1(X) \cong \pi_1(T_g)$. Then X is s-cobordant rel ∂ to $X_g = T_g \times D^2$.*

Proof. Since $H^2(X;\mathbb{Z}) \cong \mathbb{Z}$ is a direct summand of $H^2(M;\mathbb{Z})$ and is generated by cup products of classes from $H^1(X;\mathbb{Z})$ the image of $\pi_1(j_X)$ cannot be a free group. Therefore it has finite index, d say, and so $\chi(\mathrm{Im}(\pi_1(j_X))) = d\chi(F)$. Since $\mathrm{Im}(\pi_1(j_X)$ is an orientable surface group, it requires at least $2-d\chi(F) = 2(gd-d+1)$ generators. On the other hand, π needs just $2g+1$ generators. Thus if $g > 1$ we must have $d = 1$, and so $\pi_1(j_X)$ is onto. This is also clear if $g = 1$, for then $\pi_1(X) \cong H_1(X;\mathbb{Z})$ is a direct summand of $H_1(M;\mathbb{Z})$. In all cases, we may apply Theorem 4 to conclude that X is aspherical.

Any homeomorphism from ∂X to ∂X_g which preserves the product structure extends to a homotopy equivalence of pairs $(X, \partial X) \simeq (X_g, \partial X_g)$. Now $L_5(\pi_1(T_g))$ acts trivially on the s-cobordism structure set $S^s_{TOP}(X_g, \partial X_g)$, by Theorem 6.7 and Lemma 6.9 of [10]. Therefore X and X_g are TOP s-cobordant (rel ∂). $\square$

If $\pi_1(Y) \cong \mathbb{Z}$ then $\Sigma = Y \cup (T_g \times D^2)$ is 1-connected, since $\pi_1(Y)$ is generated by the image of h, and $\chi(\Sigma) = 2$. Hence Σ is a homotopy 4-sphere, containing a locally flat copy of T_g with exterior Y.

Lemma 12. *If there is a map $f : Y \to Y_g$ which extends a homeomorphism of the boundaries then Y is homeomorphic to Y_g.*

Proof. Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ be the group ring of $\pi_1(Y) = \langle t \rangle$, and let $\Pi = \pi_2(Y)$. As in Theorem 4, $H_q(Y;\Lambda) = H^q(Y;\Lambda) = 0$ for $q > 2$, and the equivariant chain complex for $\widetilde{Y}$ is chain homotopy equivalent to a finite projective Λ-complex

$$Q_* = \Pi \oplus (Z_1 \to Q_1 \to Q_0)$$

of length 2, with $Z_1 \to Q_1 \to Q_0$ a resolution of $\mathbb{Z}$. The alternating sum of the ranks of the modules Q_i is $\chi(Y) = 2g$. Hence $\Pi \cong \Lambda^{2g}$, since projective Λ-modules are free. In particular, this holds also for Y_g.

If $f : Y \to Y_g$ restricts to a homeomorphism of the boundaries then $\pi_1(f)$ is an isomorphism. Comparison of the long exact sequences of the pairs shows that f induces an isomorphism $H_4(Y, \partial Y; \mathbb{Z}) \cong H_4(Y, \partial Y; \mathbb{Z})$, and so has degree 1. Therefore $\pi_2(f) = H_2(f; \Lambda)$ is onto, by Poincaré-Lefshetz duality. Since $\pi_2(Y)$ and $\pi_2(Y_g)$ are each free of rank $2g$, it follows that $\pi_2(f)$ is an isomorphism, and so f is a homotopy equivalence, by the Whitehead and Hurewicz Theorems.

Thus f is a homotopy equivalence *rel* ∂, by the HEP, and so it determines an element of the structure set $S_{TOP}(Y_g, \partial Y_g)$. The group $L_5(\mathbb{Z})$ acts trivially on the structure set, as in Lemma 10, and so the normal invariant gives a bjection $S_{TOP}(Y_g, \partial Y_g) \cong H^2(Y_g, \partial Y_g; \mathbb{F}_2) \cong H_2(Y_g; \mathbb{F}_2)$. Since $H_2(\mathbb{Z}; \mathbb{F}_2) = 0$ the Hurewicz homomorphism maps $\pi_2(Y_g)$ onto $H_2(Y_g; \mathbb{F}_2)$. Therefore there is an $\alpha \in \pi_2(Y_g)$ whose image in $H_2(Y_g; \mathbb{F}_2)$ is the Poincaré dual of the normal invariant of f. Let f_α be the composite of the map from Y_g to $Y_g \vee S^4$ which collapses the boundary of a 4-disc in the interior of Y_g with $id_{Y_g} \vee \alpha\eta^2$, where η^2 is the generator of $\pi_4(S^2)$. Then f_α is a self homotopy equivalence of $(Y_g, \partial Y_g)$ whose normal invariant agrees with that of f. (See Theorem 16.6 of [28].) Therefore f is homotopic to a homeomorphism $Y \cong Y_g$. $\square$

However, finding such a map f to begin with seems difficult. Can we somehow use the fact that Y and Y_g are subsets of S^4? In fact, Y must be homeomorphic to Y_g if $g \geq 3$, according to [17].

Suppose now that W is an s-cobordism *rel* ∂ from X to X_g, and that $Y \cong Y_g$. Since $g \geq 1$ the 3-manifold $T_g \times S^1$ is irreducible and sufficiently large. Therefore $\pi_0(Homeo(T_g \times S^1)) \cong Out(\pi)$ [27]. If $g > 1$ then $\pi_1(T_g)$ has trivial centre, and so $Out(\pi) \cong \left(\begin{smallmatrix} Out(\pi_1(T_g)) & 0 \\ \mathbb{Z}^{2g} & \mathbb{Z}^\times \end{smallmatrix}\right)$. It follows easily that every self homeomorphism of $T_g \times S^1$ extends to a self homeomorphism of $T_g \times D^2$. Attaching $Y \times [0,1] \cong Y_g \times [0,1]$ to W along $T_g \times S^1 \times [0,1]$ gives an s-concordance from j to j_g.

If $g = 1$ then $X \cong T \times D^2$ and $Out(\pi) \cong GL(3, \mathbb{Z})$. Automorphisms of π are generated by those which may be realized by homeomorphisms of $T \times D^2$ together with those that may be realized by homeomorphisms of Y_1 [22]. Thus if embeddings of T with group $\mathbb{Z}$ are standard so are embeddings of $S^1 \times S^1 \times S^1$ with both complementary components having abelian fundamental groups.

The situation is less clear for bundles over T_g with Euler number ± 1. We may construct embeddings of such manifolds by fibre sum of an embedding of $T_g \times S^1$ with the Hopf bundle $\eta : S^3 \to S^2$. However, it is not clear how the complements change under this operation. There are natural 0-framed links representing such bundle spaces. As we saw earlier, $M(1;(1,1))$ may be obtained by 0-framed surgery on the Whitehead link. This is an interchangeable 2-component link, and so $M(1;(1,1))$ has an embedding with $X \cong Y \simeq S^1 \vee S^2$ and $\pi_1(X) \cong \pi_1(Y) \cong \mathbb{Z}$. Is this embedding characterized by these conditions? (Once again, it is enough to find a map which restricts to a homeomorphism on boundaries.)

Suppose now that F is nonorientable. We may again argue that if j is an embedding of $M(-c;(1,e))$, where $c \geq 2$, and $\pi_1(X) \cong \pi_1(\#^c RP^2)$ then X is aspherical, and hence is s-cobordant to $X_{c,e}$. Moreover, if $\pi_1(Y) = Z/2Z$ then Y is the exterior of an embedding of $\#^c RP^2$ in S^4 with normal Euler number e.

Kreck has shown that in certain cases embeddings of $\#^c RP^2$ with group $Z/2Z$ must be standard, and we should again expect that j is s-concordant to a standard embedding [18]. In particular, Kreck's result includes the case when $F = Kb$ (i.e., $c = 2$). Hence embeddings of the half-turn flat 3-manifold $M(-2;(1,0))$ and of the $\mathbb{N}il^3$-manifold $M(-2;(1,4))$ with $\pi_1(X) \cong \pi_1(Kb)$ and $\pi_1(Y) = Z/2Z$ are standard.

Seven of the thirteen 3-manifolds with elementary amenable fundamental groups that embed are total spaces of S^1-bundles (namely, S^3, S^3/Q, $S^2 \times S^1$, $S^1 \times S^1 \times S^1$, $M(-2;(1,0))$, $M(1;(1,1))$ and $M(-2;(1,4))$). Two of these (apart from S^3) and five of the others are the result of surgery on 0-framed 2-component links with trivial component knots. (See [3].) The thirteenth such 3-manifold is the Poincaré homology sphere S^3/I^*, which bounds a contractible TOP 4-manifold C (as do all homology 3-spheres) and so embeds in the double $DC \cong S^4$. However, it is well known that S^3/I^* does not embed smoothly.

Similar arguments apply to the standard embedding of $M = \#^\beta(S^2 \times S^1)$ as the boundary of a regular neighbourhood of $\vee^\beta S^1$ in S^4. If M is any closed 3-manifold with an embedding $j : M \to S^4$ for which $\pi_1(j_X)$ is an isomorphism then the natural map from $H_3(M;\mathbb{Z})$ to $H_3(\pi;\mathbb{Z})$ is 0, since it factors through $H_3(j_X) = 0$. Hence $\pi \cong F(\beta)$. Moreover, X is aspherical, by Theorem 4, and $\pi_1(Y) \cong \pi_1(X \cup_M Y) = 1$, by Van Kampen's Theorem. Arguing as in Lemma 11, we find that X is TOP s-cobordant to $\natural^\beta(D^3 \times S^1)$. Since $Y \subset S^4$, it has signature 0, and so $Y \cong \natural^\beta(S^2 \times D^2)$, by 1-connected surgery. Every self-homeomorphism of $\#^\beta(S^2 \times S^1)$ extends across $\natural^\beta(D^3 \times S^1)$, and so j is s-concordant to the standard embedding.

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